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ON THE LINEARITY OF THE LORENTZ TRANSFORMATION *

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(1) INTRODUTION.

The importance of the Lorentz transformation formulae has led to exhaustive treatment of the same by various writers. The present paper is an attempt to establish the linearity of the transformation from simple arguments. In the original deduction of the transformation Einstein [1] based the linearity on the homogeneity of space and time. Frank and Rothe [2] considered the linear homogeneous group of transformations with a single parameter (the velocity), and showed that the Lorentz transformation corresponds to the one giving the contraction of length, which must be assumed to be dependent only on the magnitude of the velocity, and not on its sign. Narliker [3] has shown that if the transformation is to form a continuous group and at the same time keeps the wave equation invariant it must necessarily be linear. G. T. Whitrow [4] in a recent paper has deduced the Lorentz transformation from the correlation of clocks of two uniformly moving observers (the *necessity* for *space-time* correlation being established), with the assumptions of the existence of an invariant velocity, of equivalence, and of the second order differentiability of the transformation function. The present treatment differs somewhat from all of these methods. For the sake of simplicity, the two dimensional problems, namely the one involving (x, t) only is considered here. The linearity is deduced from the idea of the invariance of the wave equation,

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u=0$$

in the two systems k and k', where k' has a uniform velocity v in the x-direction relative to k, and the assumption that when the velocity of k' relative to k is v, the velocity of k relative to k' is $-v$.

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(2)

The following result will physically appear to be plausible but as it is essential for our deduction a formal proof is given below.

If the wave-equation in the $k(x, t)$ system is transformed into itself in the $k'(x', t')$ system, then $(x \mp ct)$ is transformed either into $\phi(x' \mp ct')$ or $\phi(x' \mp ct')$, ϕ being a function of its argument and corresponding signs being taken in each case.

Though analytically a diverging wave may be transformed into a diverging or a converging wave, the latter possibility is ruled out from physical consideration. Let us suppose that the transformation $(x', t') \longrightarrow (x, t)$ transforms the wave equation as follows

$$\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \phi(x, t, v)\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] \quad \dots \quad (1)$$

where for generality we have taken ϕ to involve the variables also.

The substitution

$$\xi = x - ct,\ \eta = x + ct,\ \xi' = x' - ct',\ \eta' = x' + ct$$

reduces (1) to

$$\frac{\partial^2}{\partial \xi' \partial \eta'} = \phi \frac{\partial^2}{\partial \xi \partial \eta} \quad \dots \quad (2)$$

where ϕ may be easily transformed to the ξ, η variables.

Taking ξ and η to be functions of ξ' and η', we have

$$\frac{\partial}{\partial \xi'} = \frac{\partial \xi}{\partial \xi'}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial \xi'}\frac{\partial}{\partial \eta} \quad \dots \quad (3)$$

$$\frac{\partial^2}{\partial \xi' \partial \eta'} = \frac{\partial^2 \xi}{\partial \xi' \partial \eta'}\frac{\partial}{\partial \xi} + \frac{\partial^2 \eta}{\partial \xi' \partial \eta'}\frac{\partial}{\partial \eta} + \frac{\partial \eta}{\partial \eta'} \cdot \frac{\partial \eta}{\partial \xi'}\frac{\partial^2}{\partial \eta^2}$$

$$+ \frac{\partial \xi}{\partial \xi'} \cdot \frac{\partial \xi}{\partial \eta'}\frac{\partial^2}{\partial \xi^2} + \left(\frac{\partial \xi}{\partial \xi'} \cdot \frac{\partial \eta}{\partial \eta'} + \frac{\partial \xi}{\partial \eta'} \cdot \frac{\partial \eta}{\partial \xi'}\right)\frac{\partial^2}{\partial \xi \partial \eta} \quad \dots \quad (4)$$

From (2) and (4) follow

$$\frac{\partial^2 \xi}{\partial \xi' \partial \eta'} = 0, \quad \frac{\partial^2 \eta}{\partial \xi' \partial \eta'} = 0 \quad \dots \quad (5)$$

$$\frac{\partial \xi}{\partial \xi'} \cdot \frac{\partial \xi}{\partial \eta'} = \frac{\partial \eta}{\partial \xi'} \cdot \frac{\partial \eta}{\partial \eta'} = 0 \qquad \ldots \quad (6)$$

$$\frac{\partial \xi}{\partial \xi'} \cdot \frac{\partial \eta}{\partial \eta'} + \frac{\partial \xi}{\partial \eta'} \cdot \frac{\partial \eta}{\partial \xi'} \neq 0 \qquad \ldots \quad (7)$$

Equation (5) restricts ξ and η to the following form

$$\xi = f_1(\xi') + f_2(\eta'),\ \eta = F_1(\xi') + F_2(\eta') \qquad \ldots \quad (8)$$

In order that (6), (7) and (8) may hold, we must have either

$$\text{(I)} \quad \frac{\partial \xi}{\partial \eta'} = 0,\ \frac{\partial \eta}{\partial \xi'} = 0, \text{ but } \frac{\partial \xi}{\partial \xi'} \neq 0, \text{ and } \frac{\partial \eta}{\partial \eta'} \neq 0 \qquad \ldots \quad (a)$$

or

$$\text{(II)} \quad \frac{\partial \xi}{\partial \xi'} = 0,\ \frac{\partial \eta}{\partial \eta'} = 0, \text{ but } \frac{\partial \xi}{\partial \eta'} \neq 0, \text{ and } \frac{\partial \eta}{\partial \xi'} \neq 0 \qquad \ldots \quad (b)$$

The other combinations are incompatible with (7). We have from (8) by virtue of (*a*) and (*b*)

either

$$\text{(I)} \quad \xi = f_1(\xi'),\ \eta = F_2(\eta') \qquad \ldots \quad (9)$$

or

$$\text{(II)} \quad \xi = f_2(\eta'),\ \eta = F_1(\xi') \qquad \ldots \quad (10)$$

This proves the result stated above.

(3)

We now proceed to prove that the transformation is necessarily linear. Let us consider the two cases separately. When (9) holds, we put

$$\xi = x - ct = \phi((x' - ct') \qquad \ldots \quad (11)$$

$$\eta = x + ct = \psi(x' + ct') \qquad \ldots \quad (12)$$

Since the point (0, 0) in both the systems is supposed to be coincident, we have

$$\phi(0) = 0, \text{ and } \psi(0) = 0.$$

We now introduce the condition stated above thus. Firstly, we assume that the origin of k has velocity $(-v)$ with respect to the observer in the k' system, so that when $x=0$, $x' = -vt'$ for all values of t'. This gives from (11) and (12)

$$\phi\{-(c+v)t'\} = -\psi\{(c-v)t'\} \qquad \ldots \qquad \ldots \qquad \ldots \quad (13)$$

for all values of t'.

Similarly from the condition that the origin of k' has velocity v with respect to k we have, when $x'=0$, $x=vt$, whence

$$\frac{\phi\{-ct'\}}{\psi\{ct'\}}=-\frac{c-v}{c+v}. \qquad \cdots \qquad \cdots \qquad \cdots \qquad (14)$$

Putting

$$\zeta=vt', \ \gamma=ct', \ \beta=\frac{\zeta}{\gamma}=\frac{v}{c},$$

so that β is a constant (independent of t'), we have from (13) and (14)

$$\phi\{-c(1+\beta)t'\}=-\psi\{c(1-\beta)t'\} \qquad \cdots \qquad \cdots \qquad (15)$$

$$\phi(-ct')=-\frac{1-\beta}{1+\beta}\psi(ct') \qquad \cdots \qquad \cdots \qquad \cdots \qquad (16)$$

Writing (16) as

$$\phi\{-ct'(1+\beta)\}=-\frac{1-\beta}{1+\beta}\psi\{ct'(1+\beta)\} \qquad \cdots \qquad \cdots \qquad (17)$$

we have from (15) and (17)

$$\psi\{ct'(1+\beta)\}=\frac{1+\beta}{1-\beta}\psi\{ct'(1-\beta)\}$$

whence putting $c(1+\beta)=k$, $c(1-\beta)=k'$, so that $(k/k')=(c+v)/(c-v)=\alpha>1$ and $k't'=z$, we get the functional equation

$$\psi(\alpha z)=\alpha\psi(z) \qquad \cdots \qquad \cdots \qquad \cdots \qquad (18)$$

where $$\psi(0)=0 \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad (18a)$$

We have to find the solution of (18), subject to (18a).

It has been shown in this appendix that a unique solution with continuous derivative exists and is of the form

$$\psi=Az$$

where A is a constant. Thus ψ and hence also ϕ is a linear function of its argument. Since $(x-ct)$ and $(x+ct)$ are linear functions of x' and t' the linearity of the transformation is established.

Let us now consider the case when (10) holds, that is

$$x-ct=\theta(x'+ct') \qquad \dots \qquad \dots \qquad \dots \qquad \dots \quad (19)$$

$$x+ct=\chi(x'-ct') \qquad \dots \qquad \dots \qquad \dots \qquad \dots \quad (20)$$

As before, we assume that $x=0$, $x'=-vt'$ for all values of t', so that

$$x\{-(c+v)t'\}=-\theta\{(c-v)t'\} \qquad \dots \qquad \dots \qquad \dots \quad (21)$$

and again when $x'=0$, $x=vt$ for all t. Hence

$$\frac{\chi(-ct')}{\theta(ct')}=-\frac{c+v}{c-v}. \qquad \dots \qquad \dots \qquad \dots \qquad \dots \quad (22)$$

Putting

$$\lambda=vt', \mu=ct', \beta=\frac{\lambda}{\mu}=\frac{v}{c}$$

so that β is a constant, we have from (21) and (22) the following equations

$$\chi\{-c(1+\beta)t'\}=-\theta\{c(1-\beta)t'\} \qquad \dots \qquad \dots \quad (23)$$

$$\chi(-ct')=-\frac{1+\beta}{1-\beta}\theta(ct') \qquad \dots \qquad \dots \qquad \dots \quad (24)$$

for all values of t'.

Treating these equations as above we find the following functional equation

$$\theta(\alpha z)=\frac{1}{\alpha}\,\theta(z)\text{.........}(\alpha \gtreqless 1) \qquad \dots \qquad \dots \qquad \dots \quad (25)$$

with $\quad \theta(0)=0$

It can be shown that a unique solution of (25) is given by $\theta=0$. Hence also $\chi=0$. This shows that equation (10) is inconsistent with the assumption that if k' has a velocity v with respect to k, then k has a velocity $-v$ with respect to k'. The second possibility is thus physically barred out. The only allowable transformation must then be given by (9) and is a linear one.

In conclusion, I beg to acknowledge my sincere gratefulness to Prof. N. R. Sen for his kind help and guidance.

APPENDIX.

To find the solution of

$$\phi(\alpha z)=\alpha\phi(z), \qquad (\alpha>1) \qquad \ldots \qquad \text{(a)}$$

Putting $\phi(0)=0$

$$\chi(z)=\frac{d}{dz}\,\phi(z)=\varphi'(z)$$

we see that $\chi(z)$ satisfies the equation

$$\chi(z)=\chi(\alpha z), \qquad (\alpha>1) \qquad \ldots \qquad \ldots \qquad \ldots \quad \text{(b)}$$

We shall prove that χ is a constant.

Let us consider two values of z, z_1 and z_2 and put

$$Z_1=\alpha z_1,\ Z_2=\alpha z_2.$$

From (b) we have

$$\chi(Z_1)=\chi\left(\frac{Z_1}{\alpha}\right)=\chi\left(\frac{Z_1}{\alpha^2}\right)=\ldots\ldots\chi\left(\frac{Z_1}{\alpha^n}\right)$$

...

and as $n\to\infty,\ \frac{Z_1}{\alpha^n}\to 0,\ (\alpha>1).$

Hence

$$\chi(Z_1)=[\chi(Z)]_{Z\to 0}$$

Z on the right approaching zero through the values $Z_1,\ Z_1/\alpha,\ Z_1/\alpha^2 \ldots Z_1/\alpha^3 \ldots$

Similarly, we can show that

$$\chi(Z_2)=[\chi(Z)]_{Z\to 0}$$

as Z approaches zero through the values $Z_2,\ Z_2/\alpha,\ Z_2/\alpha^2, \ldots\ldots Z_2/\alpha^n \ldots\ldots$ Let us now assume that $\chi(Z)$ is continuous at the origin $Z=0$, then the two above limits on the right are the same so that

$$\chi(Z_1)=\chi(Z_2)=\ldots\ldots\ldots=\text{constant}.$$

The solution of the equations (a) is thus

$$\phi(z)=Az$$

which was to be proved.

We can deduce from above the solution of

$$\alpha\theta(\alpha z) = \theta(z), \qquad (\alpha \gtreqless 1) \ldots\ldots\ldots\ldots (c)$$

with $\theta(0) = 0$.

The differentiation of the functional equation (b) gives

$$\alpha\chi'(\alpha z) = \chi'(z).$$

Putting $\chi'(z) = \theta(z)$

we get immediately to (c). The solution of equation (b) as we have just seen is $\chi(z) = \text{const.}$, whence $\theta(z) = 0$, which also satisfies $\theta(0) = 0$. The restriction on θ is certainly satisfied if we postulate that the solution is to be a continuous function.

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